

# GENERALISED ROW AND COLUMN REMOVAL PHENOMENA AND $p$ -KOSTKA NUMBERS

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**ABSTRACT.** We explain and generalise row and column removal phenomena for Schur algebras via isomorphisms between subquotients of these algebras. In particular, we prove new reduction formulae for  $p$ -Kostka numbers and extension groups between Weyl modules and simple modules.

## 1. INTRODUCTION

This paper is concerned with the study of the representation theory of the symmetric and general linear groups over a field,  $\mathbb{k}$ , of characteristic  $p > 0$ .

Given a partition  $\lambda$  of  $n$  into at most  $d$  non-zero parts, we have associated  $\mathrm{GL}_d$ -modules:  $L(\lambda)$  the simple module of highest weight  $\lambda$ ;  $\Delta(\lambda)$  (respectively  $\nabla(\lambda)$ ) the Weyl (respectively dual Weyl) module of highest weight  $\lambda$ ; and  $I(\lambda)$  the injective cover of  $L(\lambda)$ . Applying the Schur functor to these modules, we obtain the simple modules  $D(\lambda)$  (or zero); the Specht (and dual Specht) modules  $S^\lambda$  (and  $S_\lambda$ ); and the Young modules  $Y(\lambda)$  for the symmetric group  $\mathfrak{S}_n$ .

One of the main open problems in the representation theory of general linear and symmetric groups is the following.

**Problem A:** *Given  $\lambda$  and  $\mu$  partitions of  $n$ , provide a combinatorial interpretation of the decomposition numbers  $d_{\lambda\mu} = [\nabla(\lambda) : L(\mu)]$ .*

It is well-known that Problem A is equivalent to the following (see for instance [Jam83, Theorem 3.1] and [Erd96]).

**Problem B:** *Given  $\lambda$  and  $\mu$  partitions of  $n$ , provide a combinatorial interpretation of the multiplicities  $[\mathrm{Sym}^\lambda(\mathbb{k}^d) : I(\mu)] = K_{\lambda\mu} = [\mathrm{ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}(\mathbb{k}) : Y(\mu)]$ .*

The multiplicities  $K_{\lambda\mu}$  are known as the  $p$ -Kostka numbers. Young modules, and  $p$ -Kostka numbers in particular, have been extensively studied; see for example [Erd93, Erd01, EH02, FHK08, Gil14, Gra85, Hen05, Jam83, Kl83]. In this article we prove a reduction formula for  $p$ -Kostka numbers. Let  $\lambda = (\lambda_1, \dots, \lambda_d)$  and  $\mu = (\mu_1, \dots, \mu_d)$  be partitions of  $n$ . For any fixed  $1 \leq r \leq d$ , we define partitions

$$\lambda^T = (\lambda_1, \lambda_2, \dots, \lambda_r), \quad \lambda^B = (\lambda_{r+1}, \lambda_{r+2}, \dots, \lambda_d).$$

We say that  $(\lambda, \mu)$  admits a horizontal cut (after the  $r$ th row) if  $|\lambda^T| = |\mu^T|$ . Similarly, for  $1 \leq c \leq n$ , we let

$$\lambda^L = (\lambda'_1, \lambda'_2, \dots, \lambda'_c)' \quad \lambda^R = (\lambda'_{c+1}, \dots, \lambda'_n)'.$$

We say that  $(\lambda, \mu)$  admits a vertical cut (after the  $c$ th column) if  $|\lambda^L| = |\mu^L|$ .

**Theorem 1.1.** *Let  $(\lambda, \mu)$  be a pair of partitions of  $n$  that admits a horizontal row cut. Then*

$$K_{\lambda\mu} = K_{\lambda^T\mu^T} \cdot K_{\lambda^B\mu^B}.$$

*Similarly, if  $(\lambda, \mu)$  admits a vertical cut, then*

$$K_{\lambda\mu} = K_{\lambda^L\mu^L} \cdot K_{\lambda^R\mu^R}.$$

Similar reduction formulas were previously given for (graded) decomposition numbers in [CMT02, Jam81, Don85] and for the homomorphism spaces and extension groups between Weyl and Specht modules in [FL03, LM05, Don98] and [Don98, 4.2(17)]. Our approach allows us to give a simple proof and extend all of the aforementioned (unquantised) results. For example, we also obtain new results concerning extension groups between a Weyl and a simple module, as follows.

**Theorem 1.2.** *If  $\lambda, \mu$  admit a horizontal cut, then*

$$\mathrm{Ext}_{S_{n,d}^k}^k(\Delta(\lambda), L(\mu)) = \bigoplus_{i+j=k} \mathrm{Ext}_{S_{m,r}^k}^i(\Delta(\lambda^T), L(\mu^T)) \otimes \mathrm{Ext}_{S_{n-m,d-r}^k}^j(\Delta(\lambda^B), L(\mu^B)).$$

*Similarly if  $\lambda, \mu$  admit a vertical cut, then*

$$\mathrm{Ext}_{S_{n,d}^k}^k(\Delta(\lambda), L(\mu)) = \bigoplus_{i+j=k} \mathrm{Ext}_{S_{m,r}^k}^i(\Delta(\lambda^L), L(\mu^L)) \otimes \mathrm{Ext}_{S_{n-m,d-r}^k}^j(\Delta(\lambda^R), L(\mu^R)).$$

*Here  $m$  is equal to the number of nodes above the  $r$ th row (respectively to the left of the  $c$ th column) in the partition  $\lambda$  or, equivalently, the partition  $\mu$ .*

The main idea of the proof is to construct explicit isomorphisms between subquotients of the Schur algebra; on the level of the cellular bases these simply break apart semistandard tableaux into ‘top’ and ‘bottom’ parts, in the obvious fashion.

The paper is structured as follows. In the first two sections we give a review of the construction of the Schur algebra and tensor space. The exposition here does not follow the chronological development of the theory, but is cherry-picked to be as simple and combinatorial as possible. We follow Doty–Giaquinto [DG02] for the definition of the Schur algebra via generators and relations. We also recall J. A. Green’s construction of the co-determinant basis of the Schur algebra and Murphy’s construction of an analogous basis of tensor space. In Section 4 we prove Theorem 1.1 by constructing explicit isomorphisms between subquotients of the Schur algebra and tensor space. In Section 5 we recall standard facts concerning the Schur functor and hence restate the results of Section 4 in the setting of the symmetric group.

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## 2. THE COMBINATORICS OF TENSOR SPACE

We let  $\Lambda_{n,d}$  denote the set of **compositions** of  $n$  into at most  $d$  non-zero parts. That is, the set of sequences,  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_d)$ , of non-negative integers such that the sum  $|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_d$  equals  $n$ . We let  $\Lambda_{n,d}^+ \subseteq \Lambda_{n,d}$  denote the subset consisting of the sequences  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_d)$  such that  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_d$  and refer to such sequences as **partitions**. With a partition,  $\lambda$ , is associated its **Young diagram**, which is the set of nodes

$$[\lambda] = \{(i, j) \in \mathbb{Z}_{\geq 0}^2 \mid j \leq \lambda_i\}.$$

We let  $\lambda'$  denote the **conjugate partition** obtained by flipping the Young diagram  $[\lambda]$  through the north-west to south-easterly diagonal. Given  $\lambda, \mu \in \Lambda_{n,d}^+$  we say that  $\lambda$  **dominates**  $\mu$ , and write  $\lambda \supseteq \mu$  if

$$\sum_{1 \leq i \leq r} \lambda_i \geq \sum_{1 \leq i \leq r} \mu_i$$

for all  $1 \leq r \leq d$ . There is a surjective map  $\Lambda_{n,d} \rightarrow \Lambda_{n,d}^+$  given by rearranging the rows of a composition to obtain a partition in the obvious fashion (for example if  $n = 9$  and  $d = 4$ , then  $(5, 0, 1, 3) \mapsto (5, 3, 1, 0)$ ). Under the pullback of this map we obtain the dominance ordering on the set of compositions,  $\Lambda_{n,d}$ , and we extend the notation in the obvious fashion.

Given  $\lambda \in \Lambda_{n,d}^+$  and  $\mu \in \Lambda_{n,d}$ , we define a  $\lambda$ -tableau of weight  $\mu$  to be a map  $T : [\lambda] \rightarrow \{1, \dots, d\}$  such that  $\mu_i = |\{x \in [\lambda] : T(x) = i\}|$  for  $i \geq 1$ . If  $T$  is a  $\lambda$ -tableau of weight  $\mu$ , we say that  $T$  is **semistandard** if the rows are weakly increasing from left to right and the columns are strictly increasing from top to bottom. We let  $T^\lambda$  denote the unique element of  $\text{SStd}(\lambda, \lambda)$ .

The set of all semistandard tableaux of shape  $\lambda$  and weight  $\mu$  is denoted  $\text{SStd}(\lambda, \mu)$  and we let  $\text{SStd}(\lambda, -) := \cup_{\mu \in \Lambda_{n,d}} \text{SStd}(\lambda, \mu)$ . For  $d \geq n$ , we have that  $\omega = (1^n, 0^{d-n})$  belongs to  $\Lambda_{n,d}^+$ . We refer to the tableaux of weight  $\omega$  as the set of **standard tableaux**; we let  $\text{Std}(\lambda) := \text{SStd}(\lambda, \omega)$ . We let  $t^\lambda$  denote the element of  $\text{Std}(\lambda)$  in which the first row contains the entries  $1, 2, \dots, \lambda_1$  the second row contains entries  $\lambda_1 + 1, \lambda_1 + 2, \dots, \lambda_1 + \lambda_2$  etc.

**2.1. Symmetric groups and tensor space.** Fix a pair  $n, d$  of positive integers and let  $\mathbb{k}^d$  be the  $\mathbb{k}$ -module of rank  $d$ , spanned by the column vectors,  $v_1, \dots, v_d$ , over  $\mathbb{k}$  and let  $\mathbb{T} = (\mathbb{k}^d)^{\otimes n}$ , denote the the  $n$ th tensor power of  $\mathbb{k}^d$ . The module  $\mathbb{T}$  is called **tensor space**. Tensor space has a natural basis given by the **elementary tensors** of the form

$$v_{i_1} \otimes v_{i_2} \cdots \otimes v_{i_n},$$

for some  $(i_1, i_2, \dots, i_n) \in \{1, \dots, d\}^n$ . We let  $\mathfrak{S}_{\{1, 2, \dots, n\}}$  (or simply  $\mathfrak{S}_n$ ) denote the **symmetric group** of permutations of the set  $\{1, 2, \dots, n\}$ . The symmetric group  $\mathfrak{S}_n$  acts naturally on the right of  $\mathbb{T}$ . This action is given by the place permutation of the subscripts of the elementary tensors,

$$(v_{i_1} \otimes v_{i_2} \cdots \otimes v_{i_n}) \cdot s = v_{s^{-1}(i_1)} \otimes v_{s^{-1}(i_2)} \cdots \otimes v_{s^{-1}(i_n)}.$$

and extending  $\mathbb{k}$ -linearly.

Given  $\mu \in \Lambda_{n,d}$  and  $w$  an elementary tensor in  $\mathbb{T}$ , we say that the vector  $w$  has weight  $\mu$  if  $|\{i_x \mid 1 \leq x \leq n, i_x = j\}| = \mu_j$ , for all  $j \in \{1, \dots, d\}$ . We define the  $\mu$ -weight space to be the subspace  $\mathbb{T}_\mu$  of  $\mathbb{T}$  spanned by the set of elementary tensors of weight  $\mu$ .

It is clear that the symmetric group acts by transitively permuting the set of elementary vectors of a given weight,  $\mu \in \Lambda_{n,d}$ . In particular, the elementary tensor

$$\underbrace{e_1 \otimes \cdots \otimes e_1}_{\mu_1} \otimes \underbrace{e_2 \otimes \cdots \otimes e_2}_{\mu_2} \otimes \cdots \otimes \underbrace{e_d \otimes \cdots \otimes e_d}_{\mu_d}$$

is a generator of the  $\mathfrak{S}_n$ -module  $\mathbb{T}_\mu$  and the stabiliser subgroup, denoted  $\mathfrak{S}_\mu$ , is equal to the subgroup

$$\mathfrak{S}_{\{1,2,\dots,\mu_1\}} \times \mathfrak{S}_{\{\mu_1+1,\mu_1+2,\dots,\mu_2\}} \times \cdots \times \mathfrak{S}_{\{n-\mu_d+1,n-\mu_d+2,\dots,n\}}.$$

**2.2. Murphy's basis of tensor space.** We shall now define Murphy's basis of tensor space over several steps

- Let  $\lambda \in \Lambda_{n,d}^+$  and  $\mu \in \Lambda_{n,d}$ . Given  $S \in \text{SStd}(\lambda, \mu)$  we define the row-reading element  $e_S \in \mathbb{T}$  by recording the entries of  $S$ , as read from left to right along successive rows, as the subscripts in the tensor power. For example, if

$$S = \begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 2 & \\ \hline \end{array}$$

then

$$e_S = v_1 \otimes v_1 \otimes v_3 \otimes v_2 \otimes v_2.$$

- For  $\lambda \in \Lambda_{n,d}^+$ , we have a corresponding Young subgroup  $\mathfrak{S}_\lambda$  of  $\mathfrak{S}_n$  given by the stabiliser of  $e_{T^\lambda}$ . We let  $\mathcal{O}_\lambda(e_S)$  denote the orbit sum of vectors conjugate to  $e_S$  under the natural right action of  $\mathfrak{S}_\lambda$ .
- For  $t \in \text{Std}(\lambda)$  we let  $d_t$  denote the permutation on  $n$  letters such that  $(t^\lambda)d_t = t$ .
- Given  $S \in \text{SStd}(\lambda, \mu)$  and  $t \in \text{Std}(\lambda)$ . We define

$$\rho_{St} = (\mathcal{O}_\lambda(e_S))d_t$$

**Theorem 2.1** (Murphy [Mur95]). *Tensor space  $\mathbb{T} = (\mathbb{k}^d)^{\otimes n}$  is free as a  $\mathbb{Z}$ -module with basis given by*

$$\{\rho_{Tt} \mid T \in \text{SStd}(\lambda, \mu), t \in \text{Std}(\lambda), \lambda \in \Lambda_{n,d}^+, \mu \in \Lambda_{n,d}\}.$$

**Example 2.2.** Given  $\lambda = (3, 2)$ ,  $\mu = (2, 2, 1)$  and  $S$  as above, we have that

$$\rho_{St^\lambda} = v_1 \otimes v_1 \otimes v_3 \otimes v_2 \otimes v_2 + v_1 \otimes v_3 \otimes v_1 \otimes v_2 \otimes v_2 + v_3 \otimes v_1 \otimes v_1 \otimes v_2 \otimes v_2.$$

**Example 2.3.** Tensor space  $\mathbb{T} = (\mathbb{k}^2)^{\otimes 4}$  is 16 dimensional. We have that  $\Lambda_{4,2} = \{(2, 2), (3, 1), (1, 3), (4, 0), (0, 4)\}$ . The semistandard tableaux,  $S$ ,  $T$ , and  $U$  of weight  $(2, 2)$  are as follows

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline \end{array}.$$

The standard tableaux  $\mathbf{s}_1, \mathbf{s}_2, \mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$  and  $\mathbf{u}$  are as follows

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array}.$$

The space of vectors of weight  $(2, 2)$  is 6-dimensional with basis

$$\begin{aligned} \rho_{\mathbf{s}_{\mathbf{s}_1}} &= v_1 \otimes v_1 \otimes v_2 \otimes v_2 \\ \rho_{\mathbf{s}_{\mathbf{s}_2}} &= v_1 \otimes v_2 \otimes v_1 \otimes v_2 \\ \rho_{\mathbf{T}_{\mathbf{t}_1}} &= v_1 \otimes v_1 \otimes v_2 \otimes v_2 + v_1 \otimes v_2 \otimes v_1 \otimes v_2 + v_2 \otimes v_1 \otimes v_1 \otimes v_2 \\ \rho_{\mathbf{T}_{\mathbf{t}_2}} &= v_1 \otimes v_1 \otimes v_2 \otimes v_2 + v_1 \otimes v_2 \otimes v_2 \otimes v_1 + v_2 \otimes v_1 \otimes v_2 \otimes v_1 \\ \rho_{\mathbf{T}_{\mathbf{t}_3}} &= v_1 \otimes v_2 \otimes v_1 \otimes v_2 + v_1 \otimes v_2 \otimes v_2 \otimes v_1 + v_2 \otimes v_2 \otimes v_1 \otimes v_1 \\ \rho_{\mathbf{U}_{\mathbf{u}}} &= v_1 \otimes v_1 \otimes v_2 \otimes v_2 + v_1 \otimes v_2 \otimes v_1 \otimes v_2 + v_1 \otimes v_2 \otimes v_2 \otimes v_1 \\ &\quad + v_2 \otimes v_1 \otimes v_1 \otimes v_2 + v_2 \otimes v_1 \otimes v_2 \otimes v_1 + v_2 \otimes v_2 \otimes v_1 \otimes v_1. \end{aligned}$$

### 3. THE SCHUR ALGEBRA AND THE CO-DETERMINANT BASIS

Let  $\Phi$  be the root system of type  $A_{d-1}$ :  $\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq d\}$ . Here the  $\varepsilon_i$ s form the standard orthonormal basis of the euclidean space  $\mathbb{R}^d$ . Let  $(\cdot, \cdot)$  denote the inner product on this space and define  $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ . Then  $\{\alpha_1, \dots, \alpha_{d-1}\}$  is a base of simple roots and  $\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid i < j\}$  is the corresponding set of positive roots. We let  $\alpha_i^\vee = 2\alpha_i/(\alpha_i, \alpha_i)$  for  $i = 1, \dots, d$ .

The following definition of the Schur algebra over  $\mathbb{Q}$  is due to Doty and Giaquinto [DG02, Theorem 1.4] and is very much inspired by Lusztig's modified form of the quantum universal enveloping algebra.

**Definition 3.1.** The  $\mathbb{Q}$ -algebra  $S_{n,d}^{\mathbb{Q}}$  is the associative algebra (with 1) given by generators  $1_\lambda$  ( $\lambda \in \Lambda_{n,d}$ ),  $e_{i,i+1}$ ,  $f_{i,i+1}$  ( $1 \leq i \leq d-1$ ) subject to the relations

$$(R1) \quad 1_\lambda 1_\mu = \delta_{\lambda\mu} 1_\lambda, \quad \sum_{\lambda \in \Lambda_{n,d}} 1_\lambda = 1$$

$$(R2) \quad e_{i,i+1} f_{j,j+1} - f_{j,j+1} e_{i,i+1} = \delta_{ij} \sum_{\lambda \in \Lambda_{n,d}} (\alpha_i^\vee, \lambda) 1_\lambda$$

$$(R3) \quad e_{i,i+1} 1_\lambda = \begin{cases} 1_{\lambda+\alpha_i} e_{i,i+1} & \text{if } \lambda + \alpha_i \in \Lambda_{n,d} \\ 0 & \text{otherwise} \end{cases}$$

$$(R4) \quad f_{i,i+1} 1_\lambda = \begin{cases} 1_{\lambda-\alpha_i} f_{i,i+1} & \text{if } \lambda - \alpha_i \in \Lambda_{n,d} \\ 0 & \text{otherwise} \end{cases}$$

$$(R5) \quad 1_\lambda e_{i,i+1} = \begin{cases} e_{i,i+1} 1_{\lambda-\alpha_i} & \text{if } \lambda - \alpha_i \in \Lambda_{n,d} \\ 0 & \text{otherwise} \end{cases}$$

$$(R6) \quad 1_\lambda f_{i,i+1} = \begin{cases} f_{i,i+1} 1_{\lambda+\alpha_i} & \text{if } \lambda + \alpha_i \in \Lambda_{n,d} \\ 0 & \text{otherwise} \end{cases}$$

**Remark 3.2.** It was pointed out by Rouquier (see [DG, Introduction]) that the Serre relations (R7) and (R8) as stated in [DG02, Theorem 1.4] follow from (R1) to (R6) and hence may be omitted.

**Definition 3.3.** For  $1 \leq i < j \leq d$ , we inductively define elements

$$e_{i,j} = e_{i,j-1}e_{j-1,j} - e_{j-1,j}e_{i,j-1} \quad f_{i,j} = f_{i,j-1}f_{j-1,j} - f_{j-1,j}f_{i,j-1}.$$

We define the divided powers

$$e_{i,j}^{[m]} = \frac{e_{i,j}^m}{m!} \quad f_{i,j}^{[m]} = \frac{f_{i,j}^m}{m!}$$

The integral Schur algebra  $S_{n,d}^{\mathbb{Z}}$  is the subring of  $S_{n,d}^{\mathbb{Q}}$  generated by all divided powers.

**Proposition 3.4.** We have an action of the Schur algebra  $S_{n,d}^{\mathbb{Q}}$  on  $\mathbb{T}$  defined as follows,

$$\begin{aligned} e_{i,i+1}(v_{j_1} \otimes \dots \otimes v_{j_n}) &= \sum_{\substack{1 \leq a \leq n \\ j_a = i+1}} (v_{j_1} \otimes \dots \otimes v_{j_{a-1}} \otimes \dots \otimes \dots v_{j_n}) \\ f_{i,i+1}(v_{j_1} \otimes \dots \otimes v_{j_n}) &= \sum_{\substack{1 \leq a \leq n \\ j_a = i}} (v_{j_1} \otimes \dots \otimes v_{j_{a+1}} \otimes \dots \otimes \dots v_{j_n}) \\ 1_{\lambda}(v_{j_1} \otimes v_{j_2} \otimes \dots v_{j_n}) &= \begin{cases} (v_{j_1} \otimes v_{j_2} \otimes \dots v_{j_n}) & \text{if the vector is of weight } \lambda \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

*Proof.* The relations (R1), (R3)–(R6) in Definition 3.1 are clear. We now check that (R2) holds. It is easy to see that

$$\begin{aligned} f_{i,i+1}e_{i,i+1}(v) &= \sum_{\substack{1 \leq a \leq n \\ j_a = i+1}} \left( v + \sum_{\{b \neq a \mid j_b = i\}} v_{j_1} \otimes \dots \otimes v_{j_{a-1}} \otimes \dots \otimes v_{j_{b+1}} \otimes \dots v_{j_n} \right) \\ e_{i,i+1}f_{i,i+1}(v) &= \sum_{\substack{1 \leq b \leq n \\ j_b = i}} \left( v + \sum_{\{a \neq b \mid j_a = i+1\}} v_{j_1} \otimes \dots \otimes v_{j_{a-1}} \otimes \dots \otimes v_{j_{b+1}} \otimes \dots v_{j_n} \right) \end{aligned}$$

for any  $v = (v_{j_1} \otimes \dots \otimes v_{j_n}) \in \mathbb{T}$ . It is now clear that

$$(e_{i,i+1}f_{i,i+1} - f_{i,i+1}e_{i,i+1})v = (|\{a \mid j_a = i\}| - |\{a \mid j_a = i+1\}|)v,$$

as required.  $\square$

**Definition 3.5.** Given  $1 \leq i, j \leq d$  and  $\mathsf{T} \in \text{SStd}(\lambda, \mu)$ , we let  $\mathsf{T}(i, j)$  denote the number of entries equal to  $j$  lying in the  $i$ th row of  $\mathsf{T}$ . Since  $\mathsf{T}$  is semistandard we have that  $\mathsf{T}(i, j) = 0$  for  $i > j$  and  $\sum_{1 \leq i \leq d} \mathsf{T}(i, j) = \mu_j$ .

**Definition 3.6.** Given  $S, T \in \text{SStd}(\lambda, \mu)$  we let

$$\xi_{S\lambda} = \prod_{i=d}^1 \left( \prod_{j=1}^d f_{i,j}^{[S(i,j)]} \right) \quad \xi_{\lambda T} = \prod_{i=1}^d \left( \prod_{j=1}^d e_{i,j}^{[T(i,j)]} \right)$$

(notice the ordering on these products) and we define

$$\xi_{ST} = \xi_{S\lambda} 1_\lambda \xi_{\lambda T}$$

**Example 3.7.** Let  $\lambda = (3, 3)$ ,  $\mu = (2, 2, 1, 1)$ , and  $\nu = (2, 1, 2, 1)$ . We let  $S$  and  $T$  denote the tableaux

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 2 & 4 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 3 & 3 & 4 \\ \hline \end{array},$$

respectively. We have that  $S \in \text{SStd}(\lambda, \mu)$ ,  $T \in \text{SStd}(\lambda, \nu)$ . We have that  $S(2, 4) = 1$ ,  $S(2, 2) = 2$ ,  $S(1, 3) = 1$ ,  $S(1, 1) = 2$ , and all other  $S(i, j)$  are equal to zero. Similarly,  $T(1, 2) = 1$ ,  $T(2, 3) = 2$ ,  $T(2, 4) = 1$  and all other  $T(i, j) = 0$ . Therefore,

$$\xi_{ST} = f_{1,3}^{[1]} f_{2,4}^{[1]} 1_\lambda e_{2,4}^{[1]} e_{2,3}^{[2]} e_{1,2}^{[1]}.$$

We now construct a basis of the Schur algebra over  $\mathbb{Z}$ . This basis is known as the co-determinant basis and its original construction is due to J. A. Green [Gre93]; it is generalised to the  $(q-)$ Schur algebras of more general complex reflection groups by Dipper–James–Mathas [DJM98].

**Theorem 3.8.** *The Schur algebra  $S_{n,d}^{\mathbb{Z}}$  is free as a  $\mathbb{Z}$ -module with basis*

$$\{\xi_{ST} \mid S \in \text{SStd}(\lambda, \mu), T \in \text{SStd}(\lambda, \nu) \text{ for } \lambda \in \Lambda_{n,d}^+, \mu, \nu \in \Lambda_{n,d}\}.$$

*If  $S \in \text{SStd}(\lambda, -)$ ,  $T \in \text{SStd}(\lambda, -)$  for some  $\lambda \in \Lambda_{n,d}^+$ , and  $a \in S_{n,d}^{\mathbb{Z}}$  then there exist scalars  $r(a; S, U) \in \mathbb{Z}$ , which do not depend on  $T$ , such that*

$$a\xi_{ST} = \sum_{U \in \text{SStd}(\lambda, -)} r(a; S, U) \xi_{UT} \pmod{(S_{n,d}^{\mathbb{Z}})^{\triangleright \lambda}}$$

*where  $(S_{n,d}^{\mathbb{Z}})^{\triangleright \lambda}$  is the two-sided ideal generated by the idempotent  $\sum_{\{\mu \in \Lambda_{n,d} \mid \mu \triangleright \lambda\}} 1_\mu$ . The ideal  $(S_{n,d}^{\mathbb{Z}})^{\triangleright \lambda}$  is spanned by*

$$\{\xi_{QR} \mid Q, R \in \text{SStd}(\mu, -), \mu \in \Lambda_{n,d}^+, \mu \triangleright \lambda\}.$$

*Moreover, the  $\mathbb{Z}$ -linear map  $*$  :  $S_{n,d}^{\mathbb{Z}} \rightarrow S_{n,d}^{\mathbb{Z}}$  determined by  $(\xi_{ST})^* = \xi_{TS}$ , for all  $\lambda \in \Lambda_{n,d}^+$  and all  $S, T \in \text{SStd}(\lambda, -)$ , is an anti-isomorphism of  $S_{n,d}^{\mathbb{Z}}$ . Therefore the Schur algebra is a cellular algebra in the sense of [GL96].*

*Proof.* Having established the action of Doty and Guiaquinto's presentation on tensor space, and Murphy's basis of tensor space, the above follows from [DJM98, The semistandard basis theorem].  $\square$

**Definition 3.9.** Given  $\mathbb{k}$  an algebraically closed field of characteristic  $p \geq 0$ , we define the Schur algebra  $S_{n,d}^{\mathbb{k}} := S_{n,d}^{\mathbb{Z}} \otimes \mathbb{k}$ .

**Definition 3.10.** Given  $\lambda \in \Lambda_{n,d}^+$ , we define the Weyl module  $\Delta^{\mathbb{Z}}(\lambda)$  to be the left  $S_{n,d}^{\mathbb{Z}}$ -module with basis

$$\{\xi_{\mathbf{ST}^\lambda} + (S_{n,d}^{\mathbb{Z}})^{\triangleright \lambda} \mid \mathbf{S} \in \text{SStd}(\lambda, -)\}$$

and the dual Weyl module  $\nabla^{\mathbb{Z}}(\lambda)$  to be the left  $S_{n,d}^{\mathbb{Z}}$ -module with basis

$$\{\rho_{\mathbf{St}^\lambda} + \mathbb{T}^{\triangleright \lambda} \mid \mathbf{S} \in \text{SStd}(\lambda, -)\},$$

where  $\mathbb{T}^{\triangleright \lambda}$  is left  $S_{n,d}^{\mathbb{Z}}$ -module of  $\mathbb{T}$  with basis

$$\{\rho_{\mathbf{St}} \mid \mathbf{S} \in \text{SStd}(\mu, -), \mathbf{t} \in \text{Std}(\mu), \mu \triangleright \lambda\}$$

We let  $\Delta^{\mathbb{k}}(\lambda)$  (respectively  $\nabla^{\mathbb{k}}(\lambda)$ ) denote the module  $\Delta^{\mathbb{Z}}(\lambda) \otimes_R \mathbb{k}$  (respectively  $\nabla^{\mathbb{Z}}(\lambda) \otimes_R \mathbb{k}$ ). When the context is clear, we drop the ring over which the module is defined.

**Definition 3.11.** If a module,  $M$ , has a filtration of the form

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_k = M$$

where each  $M_{i+1}/M_i$  for  $1 \leq i \leq k$  is isomorphic to some  $\Delta(\lambda^{(i)})$  (respectively  $\nabla(\lambda^{(i)})$ ) for some  $\lambda^{(i)} \in \Lambda_{n,d}^+$ , then we say that  $M$  has a  $\Delta$ - (respectively  $\nabla$ -) filtration and write  $M \in \mathcal{F}(\Delta)$  (respectively  $M \in \mathcal{F}(\nabla)$ ).

Given any  $\lambda \in \Lambda_{n,d}^+$  the Weyl module,  $\Delta(\lambda)$ , is equipped with a bilinear form  $\langle \cdot, \cdot \rangle_\lambda$  determined by

$$\xi_{\mathbf{U}\mathbf{S}}\xi_{\mathbf{T}\mathbf{V}} \equiv \langle \xi_{\mathbf{ST}^\lambda}, \xi_{\mathbf{TT}^\lambda} \rangle_\lambda \xi_{\mathbf{U},\mathbf{V}} \pmod{(S_{n,d}^{\mathbb{Z}})^{\triangleright \lambda}}$$

for  $\mathbf{S}, \mathbf{T}, \mathbf{U}, \mathbf{V} \in \text{SStd}(\lambda, -)$ . We define  $L(\lambda)$  to be the quotient of the corresponding Weyl module  $\Delta(\lambda)$  by the radical of the bilinear form  $\langle \cdot, \cdot \rangle_\lambda$ . Finally we denote by  $I(\lambda)$  the injective envelope of  $L(\lambda)$  as an  $S_{n,d}^{\mathbb{k}}$ -module.

**3.1. Generalised symmetric powers.** For  $\lambda \in \Lambda_{n,d}^+$ ,  $\mu \in \Lambda_{n,d}$  and  $\mathbf{t} \in \text{Std}(\lambda)$  let  $\mu(\mathbf{t})$  be the  $\lambda$ -tableau of weight  $\mu$  obtained from  $\mathbf{t}$  by replacing each entry  $i$  in  $\mathbf{t}$  by  $r$  if  $i$  appears in row  $r$  of  $\mathbf{t}^\mu$ . Given  $\mathbf{t} \in \text{Std}(\lambda)$ , we let  $[\mathbf{t}]_\mu$  denote the set  $\{\mathbf{s} \in \text{Std}(\lambda) \mid \mu(\mathbf{s}) = \mu(\mathbf{t})\}$ . If  $\mathbf{T} \in \text{SStd}(\lambda, \mu)$ , we write  $\mathbf{t} \in \mathbf{T}$  if  $\mu(\mathbf{t}) = \mathbf{T}$ . On the other hand, it will be convenient to say that  $\mu(\mathbf{t}) = 0$ , whenever  $\mu(\mathbf{t})$  is not semistandard. Finally, for  $\mathbf{S}, \mathbf{T} \in \text{SStd}(\lambda, \mu)$  we set

$$(7) \quad \rho_{\mathbf{ST}} := \sum_{\mathbf{t} \in \mathbf{T}} \rho_{\mathbf{St}}.$$

**Remark 3.12.** In the case that  $\mu = \omega$ , the map  $\omega : \text{Std}(\lambda) \rightarrow \text{SStd}(\lambda, \omega)$  is the bijective map which identifies standard tableaux with semistandard tableaux of weight  $\omega$ .

**Example 3.13.** Let  $n = 4$  and  $d = 2$ . Adopting the same notation as in Example 2.3 it is easy to observe that there is a unique element of  $\text{SStd}(\lambda, (2, 2))$  for each



$\lambda \in \Lambda_{4,2}^+$ . These are the tableaux  $S$ ,  $T$  and  $U$  of Example 2.3. The pullback under  $\text{Std}(\lambda) \rightarrow \text{SStd}(\lambda, (2, 2))$  is given by

$$[s_1]_{(2,2)} = \{s_1\} \quad [t_1]_{(2,2)} = \{t_1, t_2\} \quad [u]_{(2,2)} = \{u\},$$

for  $\lambda$  equal to  $(2, 2)$ ,  $(3, 1)$  and  $(4)$ , respectively. Therefore

$$\begin{aligned} \rho_{TT} = \rho_{Tt_1} + \rho_{Tt_2} = & e_1 \otimes e_2 \otimes e_1 \otimes e_2 + e_2 \otimes e_1 \otimes e_1 \otimes e_2 \\ & + e_1 \otimes e_2 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_2 \otimes e_1 + 2e_1 \otimes e_1 \otimes e_2 \otimes e_2. \end{aligned}$$

**Definition 3.14.** Given  $\mu \in \Lambda_{n,d}$ , we let

$$\text{Sym}^\mu(\mathbb{k}^d) = \text{Sym}^{\mu_1}(\mathbb{k}^d) \otimes \cdots \otimes \text{Sym}^d(\mathbb{k}^d)$$

denote the generalised symmetric power of the natural  $\text{GL}_d$ -module,  $\mathbb{k}^d$ .

**Proposition 3.15.** *The module  $\text{Sym}^\mu(\mathbb{k}^d)$  has basis given by sums of elements in the Murphy basis of tensor space of Theorem 2.1, as follows*

$$\{\rho_{ST} \mid S \in \text{SStd}(\lambda, \nu), T \in \text{SStd}(\lambda, \mu), \lambda \in \Lambda_{n,d}^+, \nu \in \Lambda_{n,d}\}.$$

*Proof.* For each  $S \in \text{SStd}(\lambda, \nu)$  and  $T \in \text{SStd}(\lambda, \mu)$  we have that  $\mathfrak{S}_\mu$  acts transitively on the set  $\{\rho_{St} \mid t \in T\}$ . Moreover, the stabiliser of any element  $\rho_{St}$  is  $\mathfrak{S}_\mu \cap d_t^{-1} \mathfrak{S}_\lambda d_t$  (see for example [Mat99, Proposition 4.4]). Therefore the element  $\rho_{ST}$  is fixed by the action of  $\mathfrak{S}_\mu$ . Hence, for every  $S \in \text{SStd}(\lambda, \nu)$  and  $T \in \text{SStd}(\lambda, \mu)$  we have that  $\rho_{ST} \in \text{Sym}^\mu(\mathbb{k}^d)$ .

The elements  $\rho_{St}$  are linearly independent and the orbits  $\{t \mid \mu(t) = T\}$  for  $T \in \text{SStd}(\lambda, \mu)$  are disjoint. Therefore the elements  $\rho_{ST}$  are linearly independent (over any field, as their coefficients in the sum in equation (7) are all 0 or 1). The result now follows from a dimension count using the formula

$$[\text{Sym}^\mu(\mathbb{k}^d) : \nabla(\lambda)] = |\text{SStd}(\lambda, \mu)|$$

and the fact that  $\nabla(\lambda)$  has basis indexed by the set  $\text{SStd}(\lambda, \nu)$ .  $\square$

**Proposition 3.16** (Section 4.8 [Gre80]). *The injective indecomposable  $S_{n,d}^{\mathbb{k}}$ -modules are precisely the indecomposable summands of  $\text{Sym}^\mu(\mathbb{k}^d)$  for  $\mu \in \Lambda_{n,d}^+$ . For  $\mu, \lambda \in \Lambda_{n,d}^+$ , we have*

$$[\text{Sym}^\mu(\mathbb{k}^d) : I(\lambda)] = K_{\mu\lambda} = \dim L_\lambda(\mu)$$

where the coefficients,  $K_{\mu\lambda}$ , are known as the  $p$ -Kostka numbers. In particular,  $[\text{Sym}^\mu(\mathbb{k}^d) : I(\lambda)] = 1$  for  $\lambda = \mu$  and 0 unless  $\mu \leq \lambda$ .

#### 4. ISOMORPHISMS BETWEEN SUBQUOTIENTS OF SCHUR ALGEBRAS

In this section, we prove the main results of this paper. In Subsection 4.1 we consider certain subsets,  $\Lambda_{n,d}^+(r, c, m) \subseteq \Lambda_{n,d}^+$ . We recall the definition of generalised row cuts on pairs of partitions and show that if  $(\lambda, \mu)$  admit such a cut and  $\lambda \triangleright \mu$ , then  $\lambda$  and  $\mu$  both belong to one of our subsets  $\Lambda_{n,d}^+(r, c, m)$ . In Subsections 4.2 to 4.5 we construct explicit isomorphisms between certain subquotients of the Schur

algebras corresponding to the sets  $\Lambda_{n,d}^+(r, c, m)$ ; all of these isomorphisms are given simply on the level of the tableaux bases.

The subquotients in which we are interested are of the following form.

**Definition 4.1.** Let  $P$  denote a partially ordered set and  $Q$  denote a subset of  $P$ . We say that  $Q$  is **saturated** if for any  $\alpha \in Q$  and  $\beta \in P$  with  $\beta \triangleleft \alpha$ , we have that  $\beta \in Q$ . We say that  $Q$  is **co-saturated** if its complement in  $P$  is saturated. If a set is saturated, co-saturated, or the intersection of a saturated and a co-saturated set, we shall say that it is **closed** under the dominance order.

**Definition 4.2.** Let  $M$  be a  $S_{n,d}^k$ -module, and  $\pi \subseteq \Lambda_{n,d}^+$  denote some closed subset under the dominance order. We say that  $M$  **belongs** to  $\pi$  if the simple composition factors of  $M$  are labelled by weights from  $\pi$ . We write  $M \in \mathcal{F}_\pi(\Delta)$  (respectively  $M \in \mathcal{F}_\pi(\nabla)$ ) if  $M$  has a  $\Delta$ -filtration (respectively  $\nabla$ -filtration) in which the  $\Delta$  (respectively  $\nabla$ ) factors are labelled by weights from  $\pi$ .

We shall use standard facts about saturated and co-saturated sets in what follows, referring to [Don98, Appendix] for more details. Much of the representation theoretic information is preserved under taking such subquotients. In Subsection 4.5 we then deduce that higher extension groups and decomposition numbers are preserved under taking generalised row and column cuts, thus simplifying the proof and extending the (unquantised) results of [Jam81, Don85, LM05] and [Don98, 4.2(17)]. In Subsection 4.6, we consider the image of the generalised symmetric powers under these functors and hence prove that  $p$ -Kostka numbers are preserved under generalised row and column removal.

**4.1. Combinatorics of partitions and generalised row cuts.** We now recall the combinatorics of generalised row cuts.

**Definition 4.3.** Given  $r, c, m \in \mathbb{N}$ , we let  $\Lambda_{n,d}(r, c, m) \subseteq \Lambda_{n,d}$  denote the set

$$\{\lambda \in \Lambda_{n,d} \mid \lambda_j \leq c \leq \lambda_i, \text{ for } 1 \leq i \leq r \text{ and } r+1 \leq j \leq d, |\lambda^T| = m\}$$

and we let

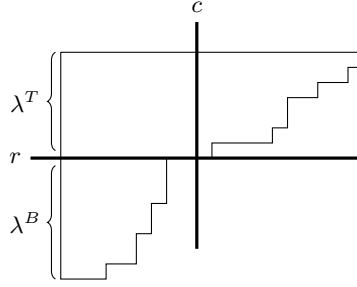
$$\Lambda_{n,d}^+(r, c, m) = \{\lambda \in \Lambda_{n,d}^+ \mid \lambda_r \geq c \geq \lambda_{r+1}, |\lambda^T| = m\}$$

in other words,  $\Lambda_{n,d}^+(r, c, m) = \Lambda_{n,d}^+ \cap \Lambda_{n,d}(r, c, m)$ . Extending the above notation we denote by  $\Lambda_{n,d}^+(0, c, 0)$  the subset of  $\Lambda_{n,d}^+$  consisting of all the partitions  $\lambda$  such that  $\lambda_1 \leq c$ .

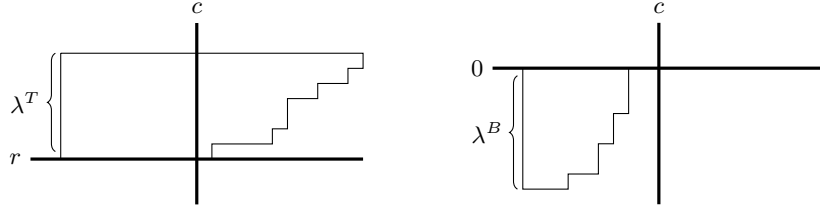
**Remark 4.4.** The subset  $\Lambda_{n,d}^+(r, c, m) \subseteq \Lambda_{n,d}^+$  can be thought of diagrammatically as in Figure 1.

**Proposition 4.5.** *The map  $\lambda \mapsto \lambda^T \times \lambda^B$  is a bijection between  $\Lambda_{n,d}(r, c, m)$  and  $\Lambda_{m,r}(r, c, m) \times \Lambda_{n-m,d-r}(0, c, 0)$ . Moreover, for  $\lambda, \mu \in \Lambda_{n,d}(r, c, m)$ , we have that  $\lambda \supseteq \mu$  if and only if  $\lambda^T \supseteq \mu^T$  and  $\lambda^B \supseteq \mu^B$ .*

*Proof.* Clear from the definitions. □

FIGURE 1. A partition  $\lambda$  such that  $\lambda_r \geq c \geq \lambda_{r+1}$ .

**Example 4.6.** For example, the map in Proposition 4.5 takes the element in Figure 1 to the pair of elements in Figure 2.

FIGURE 2. The element of  $\Lambda_{m,r}^+(r, c, m) \times \Lambda_{n-m, d-r}^+(0, c, 0)$  obtained from the element in Figure 1 under the map in Proposition 4.5.

Let  $r, c, m \in \mathbb{N}$  be such that  $\Lambda_{n,d}^+(r, c, m) \neq \emptyset$ . We note that the set  $\Lambda_{n,d}^+(r, c, m)$  has a unique maximal and a unique minimal element (under the dominance ordering on partitions). One can describe these partitions directly, however we use Proposition 4.5 to make the statements simpler. The unique maximal and minimal elements of any non-empty  $\Lambda_{z,r}^+(0, c, 0)$  are equal to

$$\alpha(r, c, z) = (c^{\lfloor \frac{z}{c} \rfloor}, z - c \lfloor \frac{z}{c} \rfloor) \quad \text{and} \quad \zeta(r, c, z) = (r^{\lfloor \frac{z}{r} \rfloor}, z - r \lfloor \frac{z}{r} \rfloor)'$$

respectively. For  $r, c \geq z$  we have that  $\alpha(r, c, z) = (z)$  and  $\zeta(r, c, z) = (1^z)$ .

**Proposition 4.7.** *If  $\Lambda_{n,d}^+(r, c, m) \neq \emptyset$ , then it has a unique maximal element*

$$\sigma := \sigma(r, c, m) = (c^r, \alpha(d - r, c, n - m)) + (m - cr)$$

*and a unique minimal element*

$$\gamma := \gamma(r, c, m) = (c^r, \zeta(d - r, c, n - m)) + \zeta(r, m - cr, m - cr).$$

*Proof.* This follows from Proposition 4.5. □

Having defined the maximal and minimal elements of  $\Lambda_{n,d}^+(r, c, m)$  we now define

$$\begin{aligned}\Sigma_{n,d}^+(r, c, m) &= \{\mu \in \Lambda_{n,d}^+ \mid \mu \leq \sigma\} \\ \Gamma_{n,d}^+(r, c, m) &= \{\mu \in \Lambda_{n,d}^+ \mid \mu \geq \gamma\}.\end{aligned}$$

The set  $\Sigma_{n,d}^+(r, c, m)$  (respectively  $\Gamma_{n,d}^+(r, c, m)$ ) is clearly a saturated (respectively co-saturated) subset of  $\Lambda_{n,d}^+$ .

We let  $\Gamma_{n,d}(r, c, m)$  (respectively  $\Sigma_{n,d}(r, c, m)$ ) denote the sets of compositions which can be obtained from a partition in  $\Gamma_{n,d}^+(r, c, m)$  (respectively  $\Sigma_{n,d}^+(r, c, m)$ ) by permutation of the rows  $\{1, \dots, r\}$  and the rows  $\{r+1, \dots, d\}$ . The sets of minimal and maximal elements of  $\Lambda_{n,d}(r, c, m)$  are those which are mapped to  $\gamma$  and  $\sigma$  respectively under the map  $\Lambda_{n,d}(r, c, m) \rightarrow \Lambda_{n,d}^+(r, c, m)$ .

**Example 4.8.** The set  $\Lambda_{10,4}^+(2, 2, 7)$  consists of two elements and is therefore equal to  $\{\sigma, \gamma\}$  where  $\sigma = (5, 2, 2, 1)$  and  $\gamma = (4, 3, 2, 1)$ .

**Example 4.9.** The set  $\Lambda_{11,5}^+(3, 2, 9)$  consists of six elements

$$(5, 2^3) \quad (4, 3, 2^2) \quad (3^3, 2) \quad (5, 2^2, 1^2) \quad (4, 3, 2, 1^2) \quad (3^3, 1^2)$$

and here we have  $\sigma = (5, 2^3)$  and  $\gamma = (3^3, 1^2)$ .

**Proposition 4.10.** *We have that*

$$\Lambda_{n,d}^+(r, c, m) = \Sigma_{n,d}^+(r, c, m) \cap \Gamma_{n,d}^+(r, c, m)$$

*Proof.* It is clear that  $\Lambda_{n,d}^+(r, c, m) \subseteq \Sigma_{n,d}^+(r, c, m) \cap \Gamma_{n,d}^+(r, c, m)$ . We now prove the reverse containment. Suppose that  $\mu \in \Lambda_{n,d}^+$  is such that  $\gamma \leq \mu \leq \sigma$ . We have that  $\sum_{1 \leq i \leq r} \gamma_i = m = \sum_{1 \leq i \leq r} \sigma_i$  and therefore

$$m \leq \sum_{1 \leq i \leq r} \mu_i \leq m.$$

Therefore  $\sum_{1 \leq i \leq r} \mu_i = m$ ; putting this together with  $\mu \leq \sigma$  and  $\sigma_r \geq c$ , we deduce that  $\mu_r \geq c$ . Similarly, we have that  $\mu \leq \sigma$  and  $\sigma_{r+1} \leq c$ ; therefore  $\mu_{r+1} \leq c$ . Therefore  $\mu \in \Lambda_{n,d}^+(r, c, m)$ , as required.  $\square$

**Definition 4.11.** Given  $\lambda, \mu \in \Lambda_{n,d}$  and  $1 \leq r \leq d$ , we say that  $\lambda$  and  $\mu$  admit a horizontal cut after the  $r$ th row if

$$\sum_{1 \leq i \leq r} \lambda_i = \sum_{1 \leq i \leq r} \mu_i.$$

**Proposition 4.12.** *Let  $\lambda, \mu \in \Lambda_{n,d}^+$  be a pair of partitions that admits a horizontal cut after the  $r$ th row. If  $\lambda \geq \mu$ , then  $\mu \in \Lambda_{n,d}^+(r, \lambda_r, |\lambda^T|)$ . Moreover  $\lambda \geq \mu$  if and only if  $\lambda^T \geq \mu^T$  and  $\lambda^B \geq \mu^B$ .*

*Proof.* Let  $\lambda, \mu \in \Lambda_{n,d}^+$  and suppose that  $\lambda$  and  $\mu$  admit a horizontal cut after the  $r$ th row and  $\lambda \geq \mu$ . In which case,

$$\mu_{r+1} \leq \lambda_{r+1} \leq \lambda_r \leq \mu_r$$

and so  $\mu \in \Lambda_{n,d}^+(r, \lambda_r, |\lambda^T|)$ . The second statement is clear.  $\square$

**4.2. Quotients of Schur algebras.** Given  $r, c, m \in \mathbb{N}$ , we have an idempotent decomposition of the identity as follows,

$$1_{\Lambda_{n,d} \setminus \Sigma_{n,d}(r,c,m)} = \sum_{\mu \notin \Sigma_{n,d}(r,c,m)} 1_\mu \quad 1_{\Sigma_{n,d}(r,c,m)} = \sum_{\mu \in \Sigma_{n,d}(r,c,m)} 1_\mu.$$

We shall consider the quotient algebras

$$S^{\mathbb{k}}(\Sigma_{n,d}(r, c, m)) := S_{n,d}^{\mathbb{k}} / (S_{n,d}^{\mathbb{k}} 1_{\Lambda_{n,d} \setminus \Sigma_{n,d}(r,c,m)} S_{n,d}^{\mathbb{k}}),$$

We have a functor  $f_{r,c,m} : S_{n,d}^{\mathbb{k}}\text{-mod} \rightarrow S^{\mathbb{k}}(\Sigma_{n,d}(r, c, m))\text{-mod}$  given by

$$f_{r,c,m}(M) = M / \langle 1_{\Lambda_{n,d} \setminus \Sigma_{n,d}(r,c,m)} M \rangle.$$

**Proposition 4.13.** *The algebra  $S^{\mathbb{k}}(\Sigma_{n,d}(r, c, m))$  is a quasi-hereditary algebra with identity  $1_{\Sigma_{n,d}(r,c,m)}$ . The algebra is free as a  $\mathbb{Z}$ -module with cellular basis*

$$\{\xi_{\mathbf{ST}} \mid \mathbf{S} \in \text{SStd}(\lambda, \mu), \mathbf{T} \in \text{SStd}(\lambda, \nu) \text{ for } \lambda \in \Sigma_{n,d}^+(r, c, m), \mu, \nu \in \Sigma_{n,d}(r, c, m)\}.$$

A full set of non-isomorphic simple, standard, and injective  $S^{\mathbb{k}}(\Sigma_{n,d}(r, c, m))$ -modules are given by

$$f_{r,c,m}(L(\lambda)) \quad f_{r,c,m}(\Delta(\lambda)) \quad f_{r,c,m}(I(\lambda))$$

respectively, for  $\lambda \in \Sigma_{n,d}^+(r, c, m)$ . If  $\lambda \notin \Sigma_{n,d}^+(r, c, m)$  we have that

$$f_{r,c,m}(L(\lambda)) = 0 \quad f_{r,c,m}(\Delta(\lambda)) = 0 \quad f_{r,c,m}(I(\lambda)) = 0.$$

Moreover, we have that

$$[\Delta(\lambda) : L(\mu)]_{S_{n,d}^{\mathbb{k}}} = [f_{r,c,m}(\Delta(\lambda)) : f_{r,c,m}(L(\mu))]_{S^{\mathbb{k}}(\Sigma_{n,d}(r,c,m))}.$$

Given  $M, N \in S_{n,d}^{\mathbb{k}}\text{-mod}$  belonging to  $\Sigma_{n,d}^+(r, c, m)$ , we have that

$$\text{Ext}_{S_{n,d}^{\mathbb{k}}}^j(M, N) \cong \text{Ext}_{S^{\mathbb{k}}(\Sigma_{n,d}(r,c,m))}^j(f_{r,c,m}(M), f_{r,c,m}(N)).$$

*Proof.* The set  $\Sigma_{n,d}(r, c, m)$  is saturated in the dominance ordering on partitions. All the results now follow from standard facts about quotient functors [Don98].  $\square$

**4.3. Subalgebras of Schur algebras.** Given  $r, c, m \in \mathbb{N}$ , we define the idempotent

$$1_{\Gamma_{n,d}^+(r,c,m)} = \sum_{\mu \in \Gamma_{n,d}^+(r,c,m)} 1_\mu$$

and associated idempotent subalgebras as follows,

$$S_{n,d}^{\mathbb{k}}(\Gamma_{n,d}^+(r, c, m)) := 1_{\Gamma_{n,d}^+(r,c,m)} S_{n,d}^{\mathbb{k}} 1_{\Gamma_{n,d}^+(r,c,m)}.$$

We have a functor  $g_{r,c,m} : S_{n,d}^{\mathbb{k}}\text{-mod} \rightarrow S_{n,d}^{\mathbb{k}}(\Gamma_{n,d}^+(r, c, m))\text{-mod}$  given by

$$g_{r,c,m}(M) = 1_{\Gamma_{n,d}^+(r,c,m)} M.$$

**Proposition 4.14.** *The algebra  $S^{\mathbb{k}}(\Gamma_{n,d}^+(r, c, m))$  is a quasi-hereditary algebra with identity  $1_{\Gamma_{n,d}^+(r, c, m)}$ . The algebra is free as a  $\mathbb{Z}$ -module with cellular basis*

$$\{\xi_{ST} \mid S \in \text{SStd}(\lambda, \mu), T \in \text{SStd}(\lambda, \nu) \text{ for } \lambda, \mu, \nu \in \Gamma_{n,d}^+(r, c, m)\}.$$

*A full set of non-isomorphic simple, standard, and injective  $S^{\mathbb{k}}(\Gamma_{n,d}^+(r, c, m))$ -modules are given by*

$$g_{r,c,m}(L(\lambda)) \quad g_{r,c,m}(\Delta(\lambda)) \quad g_{r,c,m}(I(\lambda))$$

*respectively, for  $\lambda \in \Gamma_{n,d}^+(r, c, m)$ . We have that*

$$[\Delta(\lambda) : L(\mu)]_{S_{n,d}^{\mathbb{k}}} = [g_{r,c,m}(\Delta(\lambda)) : g_{r,c,m}(L(\mu))]_{S^{\mathbb{k}}(\Gamma_{n,d}^+(r, c, m))}.$$

*Let  $N, M \in S_{n,d}^{\mathbb{k}}\text{-mod}$  and suppose that  $M \in \mathcal{F}_{\Gamma_{n,d}^+}(\Delta)$ . We have that*

$$\text{Ext}_{S_{n,d}^{\mathbb{k}}}^j(M, N) \cong \text{Ext}_{S^{\mathbb{k}}(\Gamma_{n,d}^+(r, c, m))}^j(g_{r,c,m}(M), g_{r,c,m}(N)).$$

*Proof.* The set  $\Gamma_{n,d}^+(r, c, m)$  is co-saturated in the dominance ordering on partitions. All the results now follow from standard facts about the idempotent truncation functors [Don98].  $\square$

**4.4. Subquotient algebras of Schur algebras.** Given  $r, c, m \in \mathbb{N}$ , we define the idempotent

$$1_{r,c,m}^{n,d} = \sum_{\mu \in \Lambda_{n,d}^+(r, c, m)} 1_{\mu},$$

and associated subquotient algebra

$$S^{\mathbb{k}}(\Lambda_{n,d}^+(r, c, m)) := 1_{\Gamma_{n,d}^+(r, c, m)}(S_{n,d}^{\mathbb{k}}/S_{n,d}^{\mathbb{k}}1_{\Lambda_{n,d} \setminus \Sigma_{n,d}(r, c, m)}S_{n,d}^{\mathbb{k}})1_{\Gamma_{n,d}^+(r, c, m)}.$$

We have a functor  $h_{r,c,m} : S_{n,d}^{\mathbb{k}}\text{-mod} \rightarrow S^{\mathbb{k}}(\Lambda_{n,d}^+(r, c, m))\text{-mod}$  given by

$$h_{r,c,m}(M) = 1_{\Gamma_{n,d}^+(r, c, m)}(M/\langle 1_{\Lambda_{n,d} \setminus \Sigma_{n,d}(r, c, m)}M \rangle).$$

**Proposition 4.15.** *The algebra  $S^{\mathbb{k}}(\Lambda_{n,d}^+(r, c, m))$  is a quasi-hereditary algebra with identity  $1_{r,c,m}^{n,d}$ . The algebra is free as a  $\mathbb{Z}$ -module with cellular basis*

$$\{\xi_{ST} \mid S \in \text{SStd}(\lambda, \mu), T \in \text{SStd}(\lambda, \nu) \text{ for } \lambda, \mu, \nu \in \Lambda_{n,d}^+(r, c, m)\}.$$

*A full set of non-isomorphic simple, standard, and injective  $S^{\mathbb{k}}(\Lambda_{n,d}^+(r, c, m))$ -modules are given by*

$$h_{r,c,m}(L(\lambda)) \quad h_{r,c,m}(\Delta(\lambda)) \quad h_{r,c,m}(I(\lambda))$$

*respectively, for  $\lambda \in \Lambda_{n,d}^+(r, c, m)$ . We have that*

$$[\Delta(\lambda) : L(\mu)]_{S_{n,d}^{\mathbb{k}}} = [h_{r,c,m}(\Delta(\lambda)) : h_{r,c,m}(L(\mu))]_{S^{\mathbb{k}}(\Lambda_{n,d}^+(r, c, m))}.$$

$$\mathrm{Ext}_{S_{n,d}^{\mathbf{k}}}^j(M, N) \cong \mathrm{Ext}_{S^{\mathbf{k}}(\Lambda_{n,d}^+(r,c,m))}^j(h_{r,c,m}(M), h_{r,c,m}(N)).$$

**4.5. Isomorphisms between subquotients of Schur algebras.** We now construct the isomorphism between the subquotient algebras in which we are interested. We first extend the combinatorics of cuts to semistandard tableaux.

- $S^T$  is obtained from  $S$  by deleting the  $(r+1)$ th,  $(r+2)$ th,  $\dots$  rows;
- $S^B$  is obtained from  $S$  by deleting the first  $r$  rows and replacing each entry  $i$  with the entry  $i - r$ .

Figure 1 shows three Young diagrams representing partitions of 10. The first diagram has rows of length 7, 4, 3, 2, 1, and 1. The second diagram has rows of length 6, 3, 3, 2, and 2. The third diagram has rows of length 4, 3, 2, and 1.

**Theorem 4.18.** *The map*

given by

is an isomorphism of  $\mathbb{k}$ -algebras.

$$\xi_{\lambda \Gamma} = \prod_{i=1}^d \left( \prod_{j=1}^d e_{i,j}^{[\Gamma(i,j)]} \right) = \prod_{i=r+1}^d \left( \prod_{j=r+1}^d e_{i,j}^{[\Gamma(i,j)]} \right) \times \prod_{i=1}^r \left( \prod_{j=1}^r e_{i,j}^{[\Gamma(i,j)]} \right)$$

$$= \xi_{\lambda T^B} \times \xi_{\lambda T^T}$$

and similarly for the elements  $\xi_{S\lambda}$ . Therefore, the map  $\varphi$  can be seen to be given by taking the products of generators on the left-hand side to those of the right-hand side as follows:

$$\begin{aligned} \varphi(1_\lambda) &= \begin{cases} 1_{\lambda^T} \times 1_{\lambda^B} & \text{if } \lambda \in \Lambda_{n,d}(r, c, m) \\ 0 & \text{otherwise} \end{cases} \\ \varphi(e_{i,i+1}^{[m]}) &= \begin{cases} e_{i,i+1}^{[m]} \times 1_{0,c,0}^{n-m,d-r} & \text{if } 1 \leq i \leq r-1 \\ 1_{r,c,m}^{m,r} \times e_{i-r,i-r+1}^{[m]} & \text{if } r+1 \leq i \leq d \end{cases} \\ \varphi(f_{i,i+1}^{[m]}) &= \begin{cases} f_{i,i+1}^{[m]} \times 1_{0,c,0}^{n-m,d-r} & \text{if } 1 \leq i \leq r-1 \\ 1_{r,c,m}^{m,r} \times f_{i-r,i-r+1}^{[m]} & \text{if } r+1 \leq i \leq d \end{cases}. \end{aligned}$$

and the result follows.  $\square$

We immediately obtain a new result concerning the extension groups between Weyl and simple modules.

**Corollary 4.19.** *If  $\lambda, \mu$  admit a horizontal cut after the  $r$ th row, then we have that*

$$\text{Ext}_{S_{n,d}^k}^k(\Delta(\lambda), L(\mu)) \cong \bigoplus_{i+j=k} \text{Ext}_{S_{m,r}^k}^i(\Delta(\lambda^T), L(\mu^T)) \otimes \text{Ext}_{S_{n-m,d-r}^k}^j(\Delta(\lambda^B), L(\mu^B)).$$

*Proof.* This is immediate from Proposition 4.12 and Theorem 4.18  $\square$

We also obtain the following result of Donkin [Don85].

**Corollary 4.20.** *If  $\lambda, \mu$  admit a horizontal cut after the  $r$ th row, then*

$$[\Delta(\lambda) : L(\mu)] = [\Delta(\lambda^T) : L(\mu^T)] \times [\Delta(\lambda^B) : L(\mu^B)].$$

*Proof.* This follows from Proposition 4.12 and Theorem 4.18  $\square$

We also recover the unquantised versions of [LM05] and [Don98, 4.2(17)].

**Corollary 4.21.** *If  $\lambda, \mu$  admit a horizontal cut after the  $r$ th row, then we have that*

$$\text{Ext}_{S_{n,d}^k}^k(\Delta(\lambda), \Delta(\mu)) \cong \bigoplus_{i+j=k} \text{Ext}_{S_{m,r}^k}^i(\Delta(\lambda^T), \Delta(\mu^T)) \otimes \text{Ext}_{S_{n-m,d-r}^k}^j(\Delta(\lambda^B), \Delta(\mu^B)).$$

*Proof.* This is immediate from Proposition 4.12 and Theorem 4.18  $\square$

**Remark 4.22** (Removing a single row). We now consider the example of row cuts for  $r = 1$ . In this case, the isomorphisms above (and implications for decomposition numbers and extension groups) were proven in [FHK08]. In this particularly simple case, the results can also be seen to follow by tensoring with the determinant representation and applying a duality (as noted by Donkin in [FHK08, Appendix]).



**4.6.  $p$ -Kostka numbers.** By Propositions 4.13 and 4.15, we know that injective, standard, and simple modules are all preserved under the functors  $h_{r,c,m}$  and the isomorphism  $\varphi$ . It remains to check that the generalised symmetric powers are also preserved.

**Theorem 4.23.** *Given  $\lambda, \mu \in \Lambda_{n,d}^+(r, c, m)$ , we have that*

$$h_{r,c,m}(\text{Sym}^\mu(\mathbb{k}^d)) \cong h_{r,c,m}(\text{Sym}^{\mu^T}(\mathbb{k}^r)) \otimes h_{0,c,0}(\text{Sym}^{\mu^B}(\mathbb{k}^{d-r}))$$

and

$$h_{r,c,m}(I(\lambda)) \cong h_{r,c,m}(I(\lambda^T)) \otimes h_{0,c,0}(I(\lambda^B))$$

and so we conclude that the  $p$ -Kostka numbers are preserved under generalised row cuts.

*Proof.* First, we note that  $K_{\mu\lambda} \neq 0$  implies  $\lambda \supseteq \mu$ . The isomorphism of injective modules is clear from Propositions 4.13 and 4.15 and Theorem 4.18. The result will therefore follow once we prove the isomorphism between the images of the generalised symmetric powers. Recall that the module  $\text{Sym}^\mu(\mathbb{k}^d)$  has basis

$$\{\rho_{S\mathbf{T}} \mid S \in \text{SStd}(\lambda, \nu), \mathbf{T} \in \text{SStd}(\lambda, \mu), \lambda \in \Lambda_{n,d}^+, \nu \in \Lambda_{n,d}\}.$$

Therefore  $h_{r,c,m}(\text{Sym}^\mu(\mathbb{k}^d))$  is the module with basis

$$\{\rho_{S\mathbf{T}} \mid S \in \text{SStd}(\lambda, \nu), \mathbf{T} \in \text{SStd}(\lambda, \mu), \lambda, \nu \in \Lambda_{n,d}^+(r, c, m)\}$$

and, of course, one obtains similar bases for both of the modules  $h_{r,c,m}(\text{Sym}^{\mu^T}(\mathbb{k}^r))$  and  $h_{0,c,0}(\text{Sym}^{\mu^B}(\mathbb{k}^{d-r}))$ .

Any  $\mathbf{T} \in \text{SStd}(\lambda, \mu)$  has the entry  $s$  in each of the first  $c$  columns of the  $sth$  row for each  $1 \leq s \leq r$ . Therefore, any tableau  $\mathbf{t}$  such that  $\mu(\mathbf{t}) \neq 0$  must necessarily have entries  $1, \dots, m$  in the first  $r$  rows and the entries  $m+1, \dots, n$  in the final  $d-r$  rows. Therefore, for  $\lambda, \mu \in \Lambda_{n,d}^+(r, c, m)$ , we have that the set

$$\{\mathbf{s} \in \text{Std}(\lambda) \mid \mu(\mathbf{s}) \neq 0\}$$

is naturally in bijection with the set

$$\{\mathbf{t} \in \text{Std}(\lambda^T) \mid \mu^T(\mathbf{t}) \neq 0\} \times \{\mathbf{u} \in \text{Std}(\lambda^B) \mid \mu^B(\mathbf{u}) \neq 0\},$$

via the map  $\varphi(s) = \mathbf{s}^T \times \mathbf{s}^B$ , where

- $\mathbf{s}^T$  is obtained from  $\mathbf{s}$  by deleting the  $(r+1)\text{th}$ ,  $(r+2)\text{th}$ ,  $\dots$  rows;
- $\mathbf{s}^B$  is obtained from  $\mathbf{s}$  by deleting the first  $r$  rows and replacing each entry  $i$  with the entry  $i - m$ .

Therefore, the map  $\mathbf{T} \mapsto \mathbf{T}^T \times \mathbf{T}^B$  lifts to an isomorphism

$$\psi : h_{r,c,m}(\text{Sym}^\lambda(\mathbb{k}^d)) \longrightarrow h_{r,c,m}(\text{Sym}^{\lambda^T}(\mathbb{k}^r)) \otimes h_{0,c,0}(\text{Sym}^{\lambda^B}(\mathbb{k}^{d-r}))$$

given by

$$\psi(\rho_{S\mathbf{T}}) = \psi\left(\sum_{\mathbf{t} \in \mathbf{T}} \rho_{S\mathbf{t}}\right) = \left(\sum_{\mathbf{t}^T \in \mathbf{T}^T} \rho_{S^T \mathbf{t}^T}\right) \times \left(\sum_{\mathbf{t}^B \in \mathbf{T}^B} \rho_{S^B \mathbf{t}^B}\right) = \rho_{S^T \mathbf{T}^T} \times \rho_{S^B \mathbf{T}^B}.$$

To complete the proof, it is enough to observe that if  $\lambda$  does not dominate  $\mu$  then either  $\lambda^T$  does not dominate  $\mu^T$  or  $\lambda^B$  does not dominate  $\mu^B$ , by Proposition 4.12. Hence, by Proposition 3.16, we have that  $K_{\mu\lambda} = 0 = K_{\mu^T\lambda^T} \cdot K_{\mu^B\lambda^B}$ .  $\square$

**4.7. Generalised column cuts.** Given  $\lambda \in \Lambda_{n,d}^+$  and  $1 \leq c \leq n$ , we define partitions

$$\lambda^L = (\lambda'_1, \lambda'_2, \dots, \lambda'_c)' \quad \lambda^R = (\lambda'_{c+1}, \dots, \lambda'_n)'.$$

We say that a pair of partitions  $\lambda$  and  $\mu$  admit a generalised column cut after the  $c$ th column if

$$\sum_{1 \leq i \leq c} \lambda'_i = \sum_{1 \leq i \leq c} \mu'_i$$

for some  $1 \leq c \leq n$ .

One can define similar subsets of  $\Lambda_{n,d}^+(r, c, m)$  and generalise all the arguments and isomorphisms of the previous sections to cover these reduction theorems for generalised column cuts. However, it is also easy to deduce these results in two steps as follows. If  $c = 1$ , the isomorphisms are easily deduced by tensoring with the determinant representation (plus the use of an idempotent truncation if  $d < n$ , see [FHK08] for more details). The result now follows by applying this isomorphism along with the isomorphisms of Propositions 4.13 and 4.15 and Theorem 4.18. These arguments are standard for such results, see [FL03, Proof of Proposition 2.4]. We go through this argument more explicitly for  $p$ -Kostka numbers below.

**Corollary 4.24.** *The  $p$ -Kostka numbers are preserved under generalised column cuts. In other words,  $K_{\lambda\mu} = K_{\lambda^L\mu^L} K_{\lambda^R\mu^R}$ .*

*Proof.* Suppose that  $\lambda, \mu \in \Lambda_{n,d}^+$  are such that  $\lambda \supseteq \mu$  and  $(\lambda, \mu)$  admits a vertical cut after the  $c$ th column; we let  $r = \lambda'_c$ . It is easy to see that  $(\lambda, \mu)$  admits a horizontal cut after the  $r$ th row. From tensoring with the determinant representation (see also [FHK08, Corollary 9.1]), it follows that  $p$ -Kostka numbers are preserved under first column removal. Therefore,

$$\begin{aligned} K_{\lambda\mu} &= K_{\lambda^T\mu^T} K_{\lambda^B\mu^B} \\ &= K_{(\lambda_1^T - c, \dots, \lambda_r^T - c)(\mu_1^T - c, \dots, \mu_r^T - c)} K_{\lambda^B\mu^B} \\ &= K_{\lambda^R\mu^R} K_{\lambda^B\mu^B} \\ &= K_{\lambda^R\mu^R} K_{(c^r, \lambda^B)(c^r, \mu^B)} \\ &= K_{\lambda^R\mu^R} K_{\lambda^L\mu^L} \end{aligned}$$

where the first equality follows from Theorem 4.23; the second (respectively fourth) equality follows from a total of  $c$  applications of first column removal [FHK08, Corollary 9.1] (respectively  $r$  applications of first row addition Theorem 4.23); and the third and fifth equalities follows by definition and our choice of  $r = \lambda'_c$ .  $\square$

**Remark 4.25.** In the case  $r = 1$  or  $c = 1$ , the above reduction theorems for  $p$ -Kostka numbers were first proven in [FHK08].

## 5. THE SCHUR FUNCTOR

When  $d \geq n$ , the symmetric group acts faithfully on  $1_\omega \mathbb{T}$  and we obtain an isomorphic copy of  $\mathbb{k}\mathfrak{S}_n$  as the idempotent subalgebra  $1_\omega S_{n,d}^{\mathbb{k}} 1_\omega$  of  $S_{n,d}^{\mathbb{k}}$ . In this section, we recall how one can use this idempotent truncation map (the Schur functor) to the study of the representation theory of  $\mathbb{k}\mathfrak{S}_n$ .

**5.1. The Murphy basis of the symmetric group.** Given  $\mathbf{t} \in \text{Std}(\lambda)$ , recall that  $d_{\mathbf{t}}$  is the element of  $\mathfrak{S}_n$  such that  $(\mathbf{t}^\lambda) d_{\mathbf{t}} = \mathbf{t}$ . For  $\lambda \in \Lambda_{n,d}^+$  we denote by  $x_\lambda$  the element of the group algebra of the symmetric group defined by

$$x_\lambda = \sum_{x \in \mathfrak{S}_\lambda} x.$$

**Theorem 5.1** (Murphy). *The group algebra of the symmetric group is free as a  $\mathbb{Z}$ -module with basis*

$$\{x_{\mathbf{st}} \mid x_{\mathbf{st}} := d_{\mathbf{s}} x_\lambda d_{\mathbf{t}}^{-1}, \mathbf{s}, \mathbf{t} \in \text{Std}(\lambda) \text{ for } \lambda \in \Lambda_{n,d}^+\}.$$

Recall our bijective map  $\omega : \text{Std}(\lambda) \rightarrow \text{SStd}(\lambda, \omega)$ . Suppose that  $\omega(\mathbf{s}) = \mathbf{S}$  and  $\omega(\mathbf{t}) = \mathbf{T}$ . Under this identification we obtain an isomorphism  $1_\omega S_{n,d}^{\mathbb{k}} 1_\omega \cong \mathbb{k}\mathfrak{S}_n$  given by  $\xi_{\mathbf{ST}} \mapsto x_{\mathbf{st}}$ . Therefore the basis in Theorem 5.1 is a cellular basis (in the sense of [GL96]) under the inherited cell structure (in other words, it satisfies the properties detailed in Theorem 3.8). In particular, we have the following.

**Definition 5.2.** Given  $\lambda \in \Lambda_{n,d}^+$ , we define the Specht module  $S^\lambda$  to be the left  $\mathbb{k}\mathfrak{S}_n$ -module with basis

$$\{x_{\mathbf{st}^\lambda} + \mathbb{k}\mathfrak{S}_n^{\triangleright \lambda} \mid \mathbf{s} \in \text{Std}(\lambda)\}$$

where  $\mathbb{k}\mathfrak{S}_n^{\triangleright \lambda}$  is the  $\mathbb{k}$ -submodule with basis

$$\{x_{\mathbf{uv}} \mid \mathbf{u}, \mathbf{v} \in \text{Std}(\mu), \mu \triangleright \lambda\}.$$

Similarly, we define the dual Specht module  $S_\lambda$  to be the left  $\mathbb{k}\mathfrak{S}_n$ -module with basis

$$\{\rho_{\mathbf{ST}^\lambda} + 1_\omega \mathbb{T}^{\triangleright \lambda} \mid \mathbf{S} \in \text{SStd}(\lambda, \omega)\}$$

where  $1_\omega \mathbb{T}^{\triangleright \lambda}$  is the  $\mathbb{k}$ -submodule with basis

$$\{\rho_{\mathbf{Uv}} \mid \mathbf{U} \in \text{SStd}(\mu, \omega), \mathbf{v} \in \text{Std}(\mu), \mu \triangleright \lambda\}.$$

**Definition 5.3.** We say that  $\lambda \in \Lambda_{n,d}^+$  is  $p$ -restricted if  $\lambda_i - \lambda_{i+1} < p$  for all  $1 \leq i < d$ . If  $\lambda \in \Lambda_{n,d}^+$  is not  $p$ -restricted, we say that it is  $p$ -singular.

Each Specht module  $S^\lambda$  is equipped with the bilinear form  $\langle \cdot, \cdot \rangle_\lambda$ , inherited from the idempotent truncation. This form is degenerate if and only if  $\lambda$  is  $p$ -singular. Given a  $p$ -restricted  $\lambda \in \Lambda_{n,d}^+$ , we define the simple module  $D(\lambda)$  to be the quotient of the Specht module  $S^\lambda$  by the radical of the bilinear form  $\langle \cdot, \cdot \rangle_\lambda$ . By elementary properties of idempotent truncation functors, we have that

$$[S^\lambda : D(\mu)] = [\Delta(\lambda) : L(\mu)]$$

for all  $\lambda \in \Lambda_{n,d}^+$  and all  $p$ -restricted  $\mu \in \Lambda_{n,d}^+$ . By Corollary 4.20, we have the following corollary (see also [Don85]).

**Corollary 5.4.** *Let  $\lambda$  denote a partition of  $n$  and let  $\mu$  denote a  $p$ -regular partition of  $n$ . If  $(\lambda, \mu)$  admit a horizontal cut after the  $r$ th row, then*

$$[S^\lambda : D(\mu)] = [S^{\lambda^T} : D(\mu^T)] \times [S^{\lambda^B} : D(\mu^B)].$$

*Proof.* This follows immediately from Corollary 4.20 and the above.  $\square$

**5.2. Young permutation modules.** Given  $\mu \in \Lambda_{n,d}$ , we let  $M(\mu)$  denote the image of the generalised symmetric power under the Schur functor,

$$M(\mu) = 1_\omega(\text{Sym}^\mu(\mathbb{k}^d)).$$

We refer to these modules as the **Young permutation modules**. By definition, the module  $M(\mu)$  has basis given by the subset of all the vectors of weight  $\omega$  in Proposition 3.15 as follows

$$\{\rho_{ST} \mid S \in \text{SStd}(\lambda, \omega), T \in \text{SStd}(\lambda, \mu), \lambda \in \Lambda_{n,d}^+\}.$$

Under the identification of  $\text{SStd}(\lambda, \omega)$  and  $\text{Std}(\lambda)$ , we recover Murphy's basis of these permutation modules [Mur95].

**Proposition 5.5** (J. A. Green [Gre80]). *For  $\lambda, \mu \in \Lambda_{n,d}^+$ , the module  $M(\mu)$  decomposes as a direct sum as follows*

$$M(\mu) = \bigoplus_{\lambda \vdash n} K_{\mu\lambda} Y(\lambda)$$

where  $Y(\lambda) = 1_\omega(I(\lambda))$ ; we refer to the module  $Y(\lambda)$  as the indecomposable Young module of weight  $\lambda$ .

**Corollary 5.6.** *If  $(\lambda, \mu)$  admit a horizontal cut after the  $r$ th row, then*

$$[M(\lambda) : Y(\mu)] = [M(\lambda^T) : Y(\mu^T)] \times [M(\lambda^B) : Y(\mu^B)].$$

*Proof.* This follows immediately from Theorem 4.23 and the above.  $\square$

**5.3. The faithfulness of the Schur functor.** The following theorem, proven in [KN01, Section 6.4] and [Don07, Proposition 10.5], states the degree to which cohomological information is preserved under the Schur functor.

**Theorem 5.7.** *Let  $\mathbb{k}$  denote an algebraically closed field of characteristic  $p \geq 3$ . The Schur algebra  $S_{n,d}^{\mathbb{k}}$  is a  $(p-3)$ -faithful cover (in the sense of [Rou08]) of the symmetric group,  $\mathbb{k}\mathfrak{S}_n$ . That is,*

$$\text{Ext}_{S_{n,d}^{\mathbb{k}}}^i(\Delta(\lambda), \Delta(\mu)) \cong \text{Ext}_{\mathbb{k}\mathfrak{S}_n}^i(S^\lambda, S^\mu)$$

for all  $\lambda, \mu \in \Lambda_{n,d}^+$  and all  $0 \leq i \leq p-3$ .

**Corollary 5.8.** *Let  $\mathbb{k}$  denote an algebraically closed field of characteristic  $p \geq 3$ . If  $(\lambda, \mu)$  admit a horizontal cut after the  $r$ th row, then*

$$\mathrm{Ext}_{\mathbb{k}\mathfrak{S}_n}^i(S^\lambda, S^\mu) \cong \bigoplus_{i+j=k} \mathrm{Ext}_{\mathbb{k}\mathfrak{S}_m}^i(S^{\lambda^T}, S^{\mu^T}) \otimes \mathrm{Ext}_{\mathbb{k}\mathfrak{S}_{n-m}}^j(S^{\lambda^B}, S^{\mu^B})$$

for all  $\lambda, \mu \in \Lambda_{n,d}^+$  and all  $0 \leq i \leq p-3$ .

*Proof.* This follows immediately from Corollary 4.21 and the above.  $\square$

**Remark 5.9.** This result can be partially extended to  $p = 2$ , [LM05, Theorem 1.1].

**Remark 5.10.** These results can be extended to cyclotomic Hecke algebras, [FS].

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